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# Quark-confinement mechanism for SU(2) Yang–Mills theory in abelian gauge

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**Abstract.** By dimensional reduction in the sense of Parisi and Sourlas (PS), the gauge fixing term in the abelian gauge of the SU(2) Yang–Mills field is reduced to a two-dimensional O(3) nonlinear $\sigma$ model. The confinement potential is obtained from magnetic monopoles and frame fluctuations. But the source of quark confinement is frame fluctuations and not magnetic monopoles. Because the frame $T^a$ cannot be regarded as a fixed one, the abelian projected SU(2) Yang–Mills field turns into a U(1) × U(1) gauge field – one group element being $\exp(\mathrm{i}\varphi^3 T^3)$ with fixed frame $T^3$, another group gauging the frame $T^3$. The nonperturbative part $\varpi_\mu(x)$ becomes a dynamical gauge field in two dimensions, giving rise to the short range linear potential.

## 1 Introduction

It is widely believed that the strong interaction is described by a Lagrangian density with a non-abelian SU(3) gauge theory of quarks and gluons [1,2], which is called QCD:

$$\mathcal{L} = -\frac{1}{2e^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \prod_{i=1}^{N_f}\bar{\psi}_i\left(\mathrm{i}\gamma^\mu D_\mu - M_i\right)\psi_i, \qquad (1)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon_{abc}A^a_\mu A^a_\nu$ is an SU(3) gauge field, and $\Psi_i$ are quark fields with the index $i$ labeling the flavors. Thus do appear the rich phenomena of QCD like color confinement, dynamical chiral-symmetry breaking, asymptotic freedom and quantum anomalies. In particular, confinement is the most outstanding feature in nonperturbative QCD. An isolated quark and an isolated antiquark have never been observed in experiments. Nowadays they are considered to be confined in the hadrons. This is the hypothesis of quark confinement.

In the lattice gauge theory it is easy to show that the hypothesis of quark confinement is right in the strong coupling limit. However, this result could not be continued to the weak coupling region where the string tension is expected to obey the scaling law suggested from the result of the renormalization group based on loop calculations. The first indication of [3] was based on numerical simulations within lattice gauge theory.

Although numerical evidence of quark confinement was indeed great progress towards the complete understanding of quark confinement, the analytical proof is very difficult.

One of the most important properties of QCD is asymptotic freedom, which was discovered by Gross, Wilczek and Politzer and independently by 't Hooft [4]. Using renormalization group properties (asymptotic freedom) one can express the coupling constant $e(p/\Lambda)$ at the scale $p/\Lambda$ through $\Lambda$ as

$$e^2(p/\Lambda) = \frac{-1}{b^e_0 \ln \dfrac{p^2}{\Lambda^2}}, \quad b^e_0 = \frac{11c_v}{48\pi^2} - \frac{N_f}{24\pi^2}. \qquad (2)$$

Since perturbative QCD depends on the mass scale only due to the renormalization through $\Lambda$, one can write

$$\sigma \sim m^2 = \Lambda^2 \exp\left(-\frac{1}{b^e_0 e^2}\right), \qquad (3)$$

where $\Lambda$ is the cut-off momentum. It is clear that $\sigma$ cannot be obtained from the perturbation series; hence the source of $\sigma$ and of the whole confinement phenomenon is purely nonperturbative.

Correspondingly one should admit in the QCD vacuum a nonperturbative component and split the total gluonic vector potential $A_\mu$ as follows:

$$\tilde{A}_\mu = A_\mu + \varpi_\mu, \qquad (4)$$

where $\varpi_\mu$ is the nonperturbative and $A_\mu$ the perturbative part. As for $\varpi_\mu$, it can be
(a) quasiclassical, i.e. consisting of a superposition of classical solutions like instantons, monopoles etc., or
(b) purely quantum (but nonperturbative).

It is very popular to assume that the nonperturbative contributions $\varpi_\mu$ come only from magnetic monopoles or instantons which exactly determine the low energy effective abelian gauge theory. There are examples showing that quark confinement is caused by the condensation of magnetic monopoles [5–11]. In the second picture, $\varpi_\mu$ may be a Parisi–Sourlas field in lower dimensions. For the example of a SU(2) Yang–Mills field in abelian gauge, the nonperturbative part $\varpi$ is a kind of frame connection field, a new U(1) gauge field.

In this paper it is shown that the nonperturbative contributions $\varpi_\mu$ come not only from a magnetic monopole or instanton, but also from frame fluctuations. We will calculate Wilson loops in both pictures and show their differences.

We follow the approach that Kondo has used [13] to discuss confinement of SU(2) Yang–Mills fields in a maximum abelian (MA) gauge. In the MA gauge, the physical information of the gauge configuration is concentrated in the diagonal components as much as possible. The off-diagonal gluon is minimized by the gauge transformation. Magnetic monopoles appear as topological excitations in the abelian gauge, which diagonalizes a gauge-dependent variable [6]. In the abelian gauge, SU(2) non-abelian gauge theory is reduced to a U(1) one, and monopoles appear at hedgehog-like configurations according to the nontrivial homotopy group [6,7]

$$\Pi_2\left[\mathrm{SU}(2)/\mathrm{U}(1)\right] = Z.$$

Recent lattice studies with the MA gauge have indicated monopole condensation in the Yang–Mills vacuum and the relevant role of abelian degrees of freedom (abelian dominance) [11].

It was shown that a version of an gauge fixing term in MA gauge allows us to write it in the form which is both BRST and anti-BRST exact. A hidden Osp(4 | 2) supersymmetry was found based on the superspace formulation of BRST invariant theories [14–19,13]. It turns out that the hidden supersymmetry leads to dimensional reduction in the sense of Parisi and Sourlas (PS) [14]. Consequently the MA gauge fixing term of four-dimensional SU(2) Yang–Mills fields turns into the equivalent two-dimensional O(3) nonlinear $\sigma$ model by the superspace embedding.

Based on the equivalent two-dimensional O(3) nonlinear $\sigma$ model (NLSM), we advance the confinement mechanism from frame fluctuations. When the frame $\phi^3 = U^{-1}(x)\, T^3 U(x)$ is not a fixed one in abelian projected Yang–Mills theory, the abelian projected gauge field $A_\mu(x)$ turns into $A^\phi_\mu(x) + (1/e)\varpi_\mu$, and the original abelian projected SU(2) Yang–Mills field obtains another U(1) local symmetry and turns into a U(1) × U(1) local symmetry – one group element being $\exp(\mathrm{i}\varphi^3 T^3)$ with fixed frame $T^3$, another gauging the frame $T^3$. At the same time the nonperturbative component $\varpi_\mu$ appears. In the propagator of the vector field $\varpi_\mu(x)$, a massless pole appears. Hence the frame connection field $\varpi_\mu(x)$ becomes a dynamical gauge field in two dimensions, giving rise to a confining potential. The Feynman diagram for two test particles with opposite charges $q$ scattering by exchanging one $\mathcal{W}_\mu$ shows the short range (SR) linear potential between them (massive or massless) by

$$V_{\mathrm{frame}}(r) = V_{\mathrm{SR}} \sim -\frac{q}{e}\Lambda^2 \exp\left(-\frac{1}{b_0^e e^2}\right) r. \quad (5)$$

In the limit $r \to \infty$, due to the screening effect of $z$ pairs, the long range (LR) confining potential shows a periodic behavior in $q/e$ described by

$$V_{\mathrm{LR}}(r \to \infty, q/e) = V_{\mathrm{LR}}(r \to \infty, q/e \pm 1)\,. \quad (6)$$

The contribution to the confinement potential from monopoles is easy to obtain [13]. Because the instanton configuration in two-dimensional O(3) NLSM can be identified with the magnetic monopole configuration in four dimensions, the planar Wilson loop in four-dimensional SU(2) Yang–Mills theory in MA gauge is calculated in the two-dimensional O(3) NLSM by making use of PS dimensional reduction [20]. In the limit $r \to \infty$, due to the screening effect, the confining potential is a long range one:

$$V_{\mathrm{monop}}(r) = V_{\mathrm{LR}}(r \to \infty) \sim -\left(1 - \cos\frac{2\pi q}{e}\right)\Lambda^2 \exp\left(-\frac{1}{b_0^e e^2}\right) r. \quad (7)$$

For $q = Ne$, where $N$ is any integer, the long range linear potential vanishes.

Finally, the short range linear potential $V_{\mathrm{SR}}(r)$ for quark confinement is related to $V_{\mathrm{frame}}(r)$, but on the other hand $V_{\mathrm{monop}}(r)$ to a long range linear potential which vanishes for quark and anti-quark. Hence the quark-confinement mechanism is from the frame fluctuations, not monopole condensation!

This paper is organized as follows.

In Sect. 2, the Lagrangian is constructed from a MA gauge fixing term dependent on gauge modes $g(x)$. The gauge fixing term is written in the exact form with the BRST transformation $\delta_B$ and the anti-BRST transformation $\bar{\delta}_B$.

In Sect. 3, we introduce a superspace embedding parameter $s$. From it, the relation between $s$ and $\gamma$ in superspace is obtained as $\left(4s^2/\gamma\right) = \mathrm{i}\alpha$. Through PS dimensional reduction [14], the MA gauge fixing term of the four-dimensional Yang–Mills field is reduced to the equivalent two-dimensional O(3) nonlinear $\sigma$ model. Because the coupling constant $\lambda$ for the O(3) nonlinear $\sigma$ model depends on $e^2$ and $\alpha$, we solve $\lambda$ from the known $\beta(\Lambda)$ function of $e^2$.

In Sect. 4 we introduce the confinement mechanism by frame fluctuations. In addition, the frame connection field $\varpi_\mu(x)$ can be regarded as a dynamical Parisi–Sourlas gauge field $\mathcal{W}_l\left(x, \theta, \bar{\theta}\right)$ for the reason of Osp(4|2) supersymmetry. The short range linear potential between quark and anti-quark (massive or massless) for QCD in four dimensions is obtained by exchanging one $\mathcal{W}_\mu$.

In Sect. 5 the confinement mechanism from monopole condensation is reviewed. The method developed by Kondo is similar to the calculation of the Wilson loop in the abelian Higgs model in two dimensions [20].

In Sect. 6 we draw the conclusion that the source for quark confinement via the short range linear potential $V_{\mathrm{SR}}(r)$ is frame fluctuations, not magnetic monopole condensation.

## 2 Abelian gauge fixing and abelian projection

Let us begin with the abelian gauge fixing and abelian projection.

The basic idea of abelian projection proposed by 't Hooft is to remove as many non-abelian degrees of freedom as possible, by partially fixing the gauge in such a way that there remain the local U(1) symmetry and the global Weyl symmetry [21]. Under the abelian projection, SU(2) gauge theory reduces to the $H = \mathrm{U}(1)$ abelian gauge theory plus magnetic monopoles.

For any composite field $\Phi$ transforms as an adjoint representation of the SU(2) Lie group:

$$\Phi \to \Phi' = g^{-1}\Phi g, \tag{8}$$

where $g$ (the gauge) is a specific unitary matrix. $\Phi'$ is diagonal

$$\Phi' = g^{-1}\Phi g = \mathrm{diag}(\lambda_1, \lambda_2). \tag{9}$$

For $\Phi$ from the Lie algebra of SU(2), one can choose $\lambda_1 \le \lambda_2$. It is clear that $g$ is determined up to left multiplication by a diagonal SU(2) matrix:

$$g = \begin{pmatrix} \cos\phi \mathrm{e}^{\mathrm{i}\theta} & \sin\phi \mathrm{e}^{\mathrm{i}\chi} \\ -\sin\phi \mathrm{e}^{-\mathrm{i}\chi} & \cos\phi \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}, \tag{10}$$

where

$$\chi = \Delta - \alpha/2, \quad \theta = -\Delta - \alpha/2, \quad \phi = \gamma/2.$$

Here $\alpha$ and $\gamma$ are the azimuthal and polar angles of the reference system in the given time slice; $\Delta(x)$ is an arbitrary function.

Now $A_\mu$ is transformed to the gauge

$$A_\mu^{(\Omega)} = g^{-1}\left(A_\mu + \frac{\mathrm{i}}{e}\partial_\mu\right)g, \tag{11}$$

and we will consider how the components of $A_\mu^{(\Omega)}$ transform under U(1). The diagonal ones

$$A_\mu^i \equiv (A_\mu^{(\Omega)})_{ii}, \tag{12}$$

transform as "photons":

$$A_\mu^i \to A_\mu^i = A_\mu^i + \frac{1}{e}\partial_\mu\phi_i, \tag{13}$$

while the nondiagonal ones, $A_\mu^{ij} \equiv (A_\mu^{\Omega})^{ij}$, transform as charged fields:

$$A_\mu^{'ij} = exp[i(\phi_i - \phi_j)]A_\mu^{ij}. \tag{14}$$

Actually, the choice for SU(2) is nothing but the condition of minimizing the functional $\mathcal{R}[A]$ for the gauge rotated off-diagonal gluon fields $A$, i.e., with $\min\mathcal{R}[A^{\Omega}]$,

$$\begin{aligned}\mathcal{R}[A] &= \frac{1}{2}\int \mathrm{d}^4x[(A_\mu^1(x))^2 + (A_\mu^2(x))^2] \\ &= \int \mathrm{d}^4x A_\mu^+(x)A_\mu^-(x).\end{aligned} \tag{15}$$

For SU(2), the MA gauge fixing condition is given by

$$F^{\pm}[A, a] = (\partial_\mu \pm \mathrm{i}eA_\mu^3)A_\mu^{\pm} = 0, \tag{16}$$

using the $(\pm, 3)$ basis,

$$\mathcal{O}^{\pm} = (\mathcal{O}^1 \pm \mathrm{i}\mathcal{O}^2)/\sqrt{2}. \tag{17}$$

The simplest choice of $G_{\mathrm{gf}}$ for the MA gauge in the $(\pm, 3)$ basis is given by

$$G_{\mathrm{gf}} = \sum_{\pm}\overline{\mathcal{C}}^{\pm}\left(F^{\pm}[A_\mu^{\pm}, A_\mu^3] + \frac{\alpha}{2}\phi^{\pm}\right). \tag{18}$$

Using the Faddeev–Popov formula [22], we have the path integral function

$$\begin{aligned}\mathcal{Z} &= \int DA_\mu DgD\phi D\mathcal{C}D\overline{\mathcal{C}} \\ &\times \exp\left(\mathrm{i}\int\left(-\frac{1}{2e^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_g\right)\mathrm{d}x^4\right),\end{aligned} \tag{19}$$

where the MA gauge fixing term $\mathcal{L}_g$ would be

$$\mathcal{L}_{\mathrm{gf}} = -\mathrm{tr}\left(\phi F^a[A, a] + \frac{\alpha}{2}\phi^{a2} + \bar{\mathcal{C}}^a\partial_\mu D_\mu\left[(A_\mu^{\pm})\right]\mathcal{C}^a\right), \tag{20}$$

where $a = \pm$.

The standard formulae of the BRST $\delta_B$ and anti-BRST $\bar{\delta}_B$ are

$$\begin{aligned}&\delta_B A_\mu(x) = \partial_\mu\mathcal{C}(x), \quad \delta_B\phi(x) = 0, \\ &\delta_B\mathcal{C}(x) = 0, \quad \delta_B\overline{\mathcal{C}}(x) = \mathrm{i}\phi(x), \\ &\delta_B\phi(x) = \delta_B\bar{\phi}(x) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\bar{\delta}_B A_\mu(x) = \partial_\mu\overline{\mathcal{C}}(x), \quad \bar{\delta}_B\bar{\phi}(x) = 0, \\ &\bar{\delta}_B\mathcal{C}(x) = \mathrm{i}\bar{\phi}(x), \quad \bar{\delta}_B\overline{\mathcal{C}}(x) = 0, \\ &\bar{\delta}_B\phi(x) = \bar{\delta}_B\bar{\phi}(x) = 0,\end{aligned}$$

and

$$\phi(x) + \bar{\phi}(x) = [\mathcal{C}(x), \overline{\mathcal{C}}(x)].$$

The BRST and anti-BRST transformations have the following properties:

$$(\delta_B)^2 = 0, \quad (\bar{\delta}_B)^2 = 0, \quad \{\delta_B, \bar{\delta}_B\} = \delta_B\bar{\delta}_B + \bar{\delta}_B\delta_B = 0. \tag{22}$$

The BRST $\delta_B$ and anti-BRST $\bar{\delta}_B$, $\mathcal{L}_g$ that we used turn into a very compact form:

$$\mathcal{L}_{\mathrm{gf}} = \mathrm{i}\delta_B\bar{\delta}_B\mathrm{tr}\left(\frac{1}{2}\left[\left(A_\mu^{\pm}\right)^{\Omega=g}\right]^2 + \frac{\mathrm{i}\alpha}{2}\mathcal{C}^{\pm}\overline{\mathcal{C}}^{\pm}\right). \tag{23}$$

For convenience we replace $A_\mu^{\pm}$ by $\left(A_\mu^{\pm}\right)^{\Omega=g}$ where

$$\begin{aligned}\left(A_\mu^{\pm}\right)^{\Omega=g} &= g^{-1}A_\mu^{\pm}g + \frac{1}{\mathrm{i}e}g^{-1}\partial_\mu g, \\ g(x) &\propto \mathrm{SU}(2)/\mathrm{U}(1).\end{aligned}$$

# 3 Hidden supersymmetry and superspace embedding

## 3.1 Hidden supersymmetry and Parisi–Sourlas dimensional reduction

In a (4+2)-dimensional superspace, $\left[(1/2)\left(g^{-1}\partial_\mu g +\mathrm{i}eg^{-1}A_\mu^{\pm}g\right)^2 + (\mathrm{i}\alpha)/(\mathcal{C})^{\pm}\overline{\mathcal{C}}^{\pm}\right]$ can be a contravariant supervector which transforms like the supercoordinate under Osp(4| 2). Thus a (4+2)-dimensional superspace $\mathcal{M}$ with coordinates is written

$$X^M = (x^\mu, \theta, \bar{\theta}) \in \mathcal{M}, \quad x \in R^4, \tag{24}$$

where $x^\mu$ denotes the coordinate of the four-dimensional Euclidean space, $\theta$ and $\bar{\theta}$ are anti-Hermitian Grassmann numbers. The inner product of two vectors is defined by introducing the superspace (covariant) metric tensor $\eta_{MN}$ with components

$$\eta_{\mu\nu} = \delta_{\mu\nu}, \quad \eta_{\theta\bar{\theta}} = -\eta_{\bar{\theta}\theta} = -2/\gamma, \quad \text{others} = 0. \tag{25}$$

The covariant supervector of quadratic form is

$$X^M X_M = X^M \eta_{MN} X^N = x^2 + (4/\gamma)\bar{\theta}\theta. \tag{26}$$

The supervector $\mathcal{A}_M = (\mathcal{A}_\mu, \mathcal{A}_\theta, \mathcal{A}_{\bar{\theta}})$ transforms like the supercoordinate under the orthosymplectic supergroup Osp(4| 2). The orthosymplectic supergroup Osp(4| 2) includes the four-dimensional orthogonal group O(4) which leaves $x^2$ invariant and the symplectic group Osp(2) of transformations leaving $\theta\bar{\theta}$ invariant.

The connection one-form (superspace vector potential) $\mathcal{A}(X)$ and its curvature (superspace field strength) $\mathcal{F}(X)$ in the superspace are

$$\begin{aligned}\mathcal{A}(X) &= \mathcal{A}_M(X)\mathrm{d}X^M = \mathcal{A}_\mu(x,\theta,\bar{\theta})\mathrm{d}x^\mu \\ &\quad + \mathcal{A}_\theta(x,\theta,\bar{\theta})\mathrm{d}\theta + \mathcal{A}_{\bar{\theta}}(x,\theta,\bar{\theta})\mathrm{d}\bar{\theta}, \\ \mathcal{F}(X) &= \boldsymbol{D}\mathcal{A}(X) + \frac{1}{2}[\mathcal{A}(X), \mathcal{A}(X)],\end{aligned} \tag{27}$$

where $\boldsymbol{D}$ is the exterior differential in the superspace,

$$\boldsymbol{D} = \mathrm{d} + \delta + \bar{\delta} = \frac{\partial}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial}{\partial\theta}\mathrm{d}\theta + \frac{\partial}{\partial\bar{\theta}}\mathrm{d}\bar{\theta}. \tag{28}$$

Setting

$$\mathcal{F}(X) = \frac{1}{2}\mathcal{F}_{\mu\nu}(X)\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{29}$$

we have the horizontal condition

$$\mathcal{F}_{M\theta}(X) = \mathcal{F}_{M\bar{\theta}}(X) = 0. \tag{30}$$

From the horizontal condition, the dependence of the superfield $A_M(x,\theta,\bar{\theta})$ on $\theta, \bar{\theta}$ is determined as follows:

$$\begin{aligned}&\partial_\theta\mathcal{A}_\mu(X) = \partial_\mu\mathcal{A}_\theta(X) - \mathrm{i}[\mathcal{A}_\mu(X), \mathcal{A}_\theta(X)], \\ &\partial_\theta\mathcal{A}_\theta(X) = \mathrm{i}\frac{1}{2}[\mathcal{A}_\theta(X), \mathcal{A}_\theta(X)], \\ &\partial_{\bar{\theta}}\mathcal{A}_\mu(X) = \partial_\mu\mathcal{A}_{\bar{\theta}}(X) - \mathrm{i}[\mathcal{A}_\mu(X), \mathcal{A}_{\bar{\theta}}(X)], \\ &\partial_{\bar{\theta}}\mathcal{A}_{\bar{\theta}}(X) = \mathrm{i}\frac{1}{2}[\mathcal{A}_{\bar{\theta}}(X), \mathcal{A}_{\bar{\theta}}(X)], \\ &\partial_\theta\mathcal{A}_{\bar{\theta}}(X) + \partial_{\bar{\theta}}\mathcal{A}_\theta(X) = -\{\mathcal{A}_\theta(X), \mathcal{A}_{\bar{\theta}}(X)\}.\end{aligned} \tag{31}$$

For the components which cannot be determined by the horizontal condition alone, we use the following identification:

$$\partial_\theta\mathcal{A}_{\bar{\theta}}(x,0,0) = \mathrm{i}\Theta(x), \quad \partial_{\bar{\theta}}\mathcal{A}_\theta(x,0,0) = \mathrm{i}\bar{\Theta}(x). \tag{32}$$

This corresponds to $\mathcal{F}_{\theta\bar{\theta}} = 0$ and gives

$$\mathrm{i}\Theta(x) + \mathrm{i}\bar{\Theta}(x) + \{\mathcal{A}_\theta(x), \mathcal{A}_{\bar{\theta}}(x)\} = 0. \tag{33}$$

From these results, it turns out that the derivatives in the direction of $\theta, \bar{\theta}$ give respectively the BRST and the anti-BRST transformations,

$$\frac{\partial}{\partial\theta} = s\delta_B, \quad \frac{\partial}{\partial\bar{\theta}} = s\bar{\delta}_B. \tag{34}$$

At the same time, if we require that

$$\mathcal{A}_\mu(X)\mid_{\theta=0,\bar{\theta}=0} = \mathcal{A}_\mu(x,0,0) = g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^{\pm}g, \tag{35}$$

we find "the superspace embedding relation" to be

$$\begin{aligned}\mathcal{A}_\theta(x) &= s\mathcal{C}^{\pm}(x), \quad \mathcal{A}_{\bar{\theta}}(x) = s\overline{\mathcal{C}}^{\pm}(x), \\ \Theta(x) &= s^2\phi(x), \quad \bar{\Theta}(x) = s^2\bar{\phi}(x),\end{aligned} \tag{36}$$

where $s$ is a c-number.

Then the inner product of $\mathcal{A}(X)$ is

$$\begin{aligned}&\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X) = \mathcal{A}^\mu(X)\mathcal{A}_\mu(X) + (2/\gamma) \\ &\times\left[\mathcal{A}_\theta(X)\mathcal{A}_{\bar{\theta}}(X) + \mathcal{A}_{\bar{\theta}}(X)\mathcal{A}_\theta(X)\right],\end{aligned} \tag{37}$$

which is invariant under superrotations. From "the superspace embedding relation", we have

$$\begin{aligned}\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X) &= \mathcal{A}^\mu(X)\mathcal{A}_\mu(X) \\ &+ \left(\frac{4s^2}{\gamma}\right)\mathcal{C}(X)\overline{\mathcal{C}}(X), \\ &= (\mathcal{A}_\mu(X))^2 + \mathrm{i}\alpha\mathcal{C}(X)\overline{\mathcal{C}}(X).\end{aligned} \tag{38}$$

Then the relation between $s$ and $\gamma$ in superspace is obtained:

$$\left(\frac{4s^2}{\gamma}\right) = \mathrm{i}\alpha. \tag{39}$$

The operator

$$\mathcal{O}(x) = \mathrm{itr}\left[\frac{1}{2}\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^{\pm}g\right)^2 + \frac{\mathrm{i}\alpha}{2}\mathcal{C}^{\pm}(x)\bar{\mathcal{C}}^{\pm}(x)\right] \tag{40}$$

has the corresponding superfield given by

$$\begin{aligned}\mathcal{O}(X) = &\frac{\mathrm{i}}{2}\mathrm{tr}\left((\mathcal{A}_\mu(X))^2\right. \\ &\left.+\ (2/\gamma)\left[\mathcal{A}_\theta(X)\mathcal{A}_{\bar{\theta}}(X) + \mathcal{A}_{\bar{\theta}}(X)\mathcal{A}_\theta(X)\right]\right).\end{aligned} \tag{41}$$

Then the superfield $\mathcal{O}(X)$ is written in Osp(4| 2) invariant form as

$$\mathcal{O}(X) = \frac{\mathrm{i}}{2}\mathrm{tr}\left(\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X)\right). \tag{42}$$

For the Grassmann number, the integration $\int \mathrm{d}\theta \left(\int \mathrm{d}\bar{\theta}\right)$ is equivalent to the differentiation $\mathrm{d}/\mathrm{d}\theta \left(\mathrm{d}/\mathrm{d}\bar{\theta}\right)$. Hence the BRST $\delta_B$ and anti-BRST $\delta_B$ transformation have the following correspondence:

$$s\delta_B \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\theta} \leftrightarrow \int \mathrm{d}\theta, \tag{43}$$
$$s\bar{\delta}_B \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}} \leftrightarrow \int \mathrm{d}\bar{\theta}.$$

This implies

$$\int \mathrm{d}\theta \mathrm{d}\bar{\theta} \mathcal{O}(x,\theta,\bar{\theta}) = -\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\mathcal{O}(x,\theta,\bar{\theta}) \tag{44}$$
$$= -s^2\bar{\delta}_B\delta_B\mathcal{O}(x) = s^2\delta_B\bar{\delta}_B\mathcal{O}(x).$$

Thus the action can be written in the manifestly superspace covariant form:

$$S_{\mathrm{gf}} = \frac{\mathrm{i}}{2s^2}\int \mathrm{d}^4x \int \mathrm{d}\theta\mathrm{d}\bar{\theta}\,\mathrm{tr}(\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X)). \tag{45}$$

Let us split the four-dimensional Euclidean space into two subsets,

$$x = (z,\hat{x}) \in R^4, \quad z \in R^2, \quad \hat{x} \in R^2.$$

Hence, for the supersymmetric operator $\mathcal{O}(X)$ one obtains

$$\mathcal{O}(x,\theta,\bar{\theta}) = f(z,\hat{x}^2) + \frac{4}{\gamma}\bar{\theta}\theta\frac{\mathrm{d}}{\mathrm{d}\hat{x}^2}f(z,\hat{x}^2). \tag{46}$$

Therefore, for the gauge fixing action of the supersymmetric model we find

$$S_{\mathrm{gf}} = \frac{1}{s^2}\int \mathrm{d}^4x \int \mathrm{d}\theta\mathrm{d}\bar{\theta}\mathcal{O}(x,\theta,\bar{\theta}) \tag{47}$$
$$= \frac{1}{s^2}\int \mathrm{d}^2z \int \mathrm{d}^2\hat{x} \int \mathrm{d}\theta \int \mathrm{d}\bar{\theta}\frac{4}{\gamma}\bar{\theta}\theta\frac{\mathrm{d}}{\mathrm{d}\hat{x}^2}f(z,\hat{x}^2)$$
$$= \frac{-4}{s^2\gamma}\int \mathrm{d}^2z \int \mathrm{d}^2\hat{x}\frac{\mathrm{d}}{\mathrm{d}\hat{x}^2}f(z,\hat{x}^2)$$
$$= \frac{4\pi}{s^2\gamma}\int \mathrm{d}^2z\mathcal{O}_0((z,0),0,0),$$

where we have assumed $f(z,\infty) \equiv \mathcal{O}_0((z,\infty),0,0) = 0$.

Thus the four-dimensional MA gauge fixing term of the Yang–Mills field is reduced to the two-dimensional SU(2)/U(1) = O(3) nonlinear $\sigma$ model (NLSM). In our model, the operator is

$$\mathcal{O}_0((z,0),0,0) = \left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^{\pm}g\right)^2 + \mathrm{i}\alpha\mathcal{C}^{\pm}(x)\overline{\mathcal{C}}^{\pm}(x)\right]. \tag{48}$$

Then the action is reduced to a two-dimensional nonlinear sigma model by

$$S_{\mathrm{gf}} = \frac{\mathrm{i}}{2s^2}\int \mathrm{d}^4x \int \mathrm{d}\theta\mathrm{d}\bar{\theta}\,\mathrm{tr}(\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X))$$
$$= -\frac{1}{2\lambda}\int \mathrm{d}x^2\mathrm{tr}\left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^{\pm}g\right)^2 + \mathrm{i}\alpha\mathcal{C}^{\pm}(x)\overline{\mathcal{C}}^{\pm}(x)\right], \tag{49}$$

where $\lambda = -s^4e^2/(\pi\alpha)$.

### 3.2 One-loop superspace embedding parameter

The superspace embedding factor $s\,(p)$ is a key parameter to bridge the original Yang–Mills field $A_\mu$ and the gauge modes $g\,(x)$ in the sigma model. But it is still unknown. Because the coupling constant $\lambda$ for the O(3) nonlinear $\sigma$ model is dependent on $e^2$, $\alpha$ and $s$, we can solve $s$ from the known $\beta\,(p/\Lambda)$ function of $e^2$, $\alpha$ and $\lambda$ [4].

To approach the O(3) nonlinear $\sigma$ model [23,24], the $g$ field is also divided into two parts [25]: one is the slowly varying vector $g_0$; the other the fast one $\xi\,(x)$;

$$g = g_0\exp\left[\mathrm{i}\lambda\xi\,(x)\right]. \tag{50}$$

The $\xi\,(x)$ are Goldstone modes which obtain a mass gap $m^2$ in the disorder phase. By $g_0$ and $\xi\,(x)$, we can rewrite the action

$$S_{\mathrm{gf}} = -\frac{1}{2\lambda}\int \mathrm{d}x^2\mathrm{tr}\left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^{\pm}g\right)^2 + \mathrm{i}\alpha\mathcal{C}^{\pm}(x)\overline{\mathcal{C}}^{\pm}(x)\right], \tag{51}$$

to

$$S_{\mathrm{gf}} = S\left(g_0, A_\mu^{\pm}\right) + \int \mathrm{d}x^2\mathrm{tr}\left\{(\partial_\mu\xi)^2 + \frac{1}{2}\left(g_0^{-1}\partial_\mu g_0 + \mathrm{i}eg_0^{-1}A_\mu^{\pm}g_0\right)[\xi,\partial_\mu\xi]\right\}. \tag{52}$$

Note that $\left(g_0^{-1}\partial_\mu g_0 + \mathrm{i}eg_0^{-1}A_\mu^{\pm}g_0\right)$ plays the role of a background field. In two dimensions, the one-loop effective action is written as

$$\mathrm{tr}\left(g_0^{-1}\partial_\mu g_0 + \mathrm{i}eg_0^{-1}A_\mu^{\pm}g_0\right)^2\left(\frac{1}{4\pi}\ln\frac{\Lambda^2}{p^2}\right). \tag{53}$$

The $\beta\,(\lambda)$ function of $\lambda$ is given by

$$\beta\,(\lambda) = \Lambda\frac{\partial\lambda}{\partial\Lambda} = -b_0^{\lambda}\lambda^2, \tag{54}$$

with $b_0^{\lambda} = 1/(4\pi)$.

To calculate the $\beta\left(e^2\right)$ function of $e^2$, one must consider the off-diagonal gluons. It was shown that the off-diagonal gluons renormalize the effective abelian gauge theory and let the coupling constant of effective abelian gauge theory run according to the renormalization group $\beta$ function which is exactly the same as the original QCD, at least up to one loop [26]. Then the $\beta\left(e^2\right)$ function of $e^2$ is

$$\beta\left(e^2\right) = -b_0^e e^4, \tag{55}$$
$$b_0^e = \frac{11c_v}{48\pi^2} - \frac{N_f}{24\pi^2},$$

where $N_f$ is the flavor number of fermions.

Up to one loop in the high energy limit $p^2 \to \infty$, the renormalization behaviors of the known parameters is described by

$$\lambda\,(p\to\infty) = \frac{-1}{b_0^{\lambda}\ln\left(p^2/\Lambda^2\right)}, \tag{56}$$
$$e^2\,(p\to\infty) = \frac{-1}{b_0^{e}\ln\left(p^2/\Lambda^2\right)},$$

where $\Lambda$ is an energy cut-off. The solution of $s$ of one loop is

$$s = \left(-\frac{\pi\alpha\lambda}{e^2}\right)^{1/4}\Bigg|_{p\to\infty} = \left(-\frac{\pi\alpha b_0^e}{b_0^\lambda}\right)^{1/4}. \quad (57)$$

Using the solution we can keep the coupling constant in the form

$$\lambda = -\frac{s^4 e^2}{\pi\alpha} = \frac{b_0^e}{b_0^\lambda}e^2 = \frac{11}{6\pi}\left(1-\frac{N_f}{11}\right)e^2, \quad (58)$$

which is independent on the gauge parameter $\alpha$ [1].

For four-dimensional Yang–Mills theory the gauge fixing term is reduced to

$$S_{\text{gf}} = -\frac{1}{2\lambda}\int \mathrm{d}x^2 \mathrm{tr}\left[\left(g^{-1}\partial_\mu g + eg^{-1}A_\mu^{\pm}g\right)^2 + \mathrm{i}\alpha\mathcal{C}^{\pm}(x)\overline{\mathcal{C}}^{\pm}(x)\right]. \quad (59)$$

The SU(2) Yang–Mills case with asymptotic freedom can be reduced to the corresponding two-dimensional O(3) nonlinear $\sigma$ model. In the two-dimensional O(3) nonlinear $\sigma$ model one has only one phase: a disorder phase without long range order,

$$\langle g\rangle\,|_{2D} = 0. \quad (60)$$

In this phase the two branches of the Goldstone excitons are massive. Up to one loop, the mass gap is obtained:

$$m^2 = \Lambda^2\exp\left(-\frac{4\pi}{\lambda}\right) = \Lambda^2\exp\left(-\frac{1}{b_0^e e^2}\right), \quad (61)$$

where $\Lambda$ is a cut-off.

## 4 Source of confinement from frame fluctuations

### 4.1 Gauge theory without a fixed frame

To discuss the confinement mechanism as originating from frame fluctuations we will need to learn about the concept of frame fluctuations. We have considered an abelian projected Yang–Mills theory with fixed frame $T^3$. In the disorder phase of the corresponding two-dimensional O(3) nonlinear $\sigma$ model, frame fluctuations are induced by quantum fluctuations of the gauge modes. The frame cannot be regarded as a fixed one $T^3$. We introduce the frame field $\phi^3(x)$ to indicate the color-direction variable. $\phi^3(x)$ is a composite field of the gauge modes $g(x)$.

We require the following properties for the frame field $\phi^3(x)$ in the gauge field. The frame field $\boldsymbol{\phi}(x)$ directly corresponds to the gauge function $U(x)\in \mathrm{SU}(2)/\mathrm{U}(1)$ via $\phi^3(x) = U^{-1}(x)T^3U(x)$. The frame field $\phi^3(x)$ consists of the three components $\phi_j(x)$ $(j=1,2,3)$ of SU(2). Each $\phi_j$ is defined to be a hermite composite scalar $\phi_j(x) = \phi_j^3 T^3$ with $\phi_j^a \in \mathbf{R}$. Corresponding to the direct product of SU(2), the three components $\phi_j(x)$ are to commute with each other: $[\phi_i(x),\phi_j(x)] = 0$, and they are normalized as $\mathrm{tr}\{\phi_i(x)\phi_j(x)\} = (1/2)\delta_{ij}$.

Without a fixed frame, the gauge field

$$A_\mu(x) = A_\mu^3(x)T^3 \quad (62)$$

changes its path-integral form. Let us obtain it through the following path-integral function:

$$W[A] = \frac{1}{\mathcal{N}}\mathrm{tr}\left[\mathcal{P}\exp\left(\mathrm{i}e\int_0^x A_\mu \mathrm{d}x_\mu\right)\right]. \quad (63)$$

With the vacuum state

$$|\phi(x),\Lambda\rangle = U(x)|\Lambda\rangle, \quad U(x)\in \mathrm{SU}(2)/\mathrm{U}(1), \quad (64)$$

we replace the trace with

$$\mathrm{tr}(\cdots) = \int D\mu(\phi)\langle\phi_N,\Lambda|(\cdots)|\phi_N,\Lambda\rangle. \quad (65)$$

The path-integral function is defined as the trace of the path-ordered exponent along the closed loop $C$, where $\mathcal{N}$ is the dimension of the representation, taken from 0 and ending at $x$. The interval $x$ can be divided into $N$ infinitesimal steps in the limit $N\to\infty$, $\delta x\to 0$ with $N\delta x = x$ being kept fixed,

$$W[A] = \lim_{N\to\infty,\delta x\to 0}\mathrm{tr}\prod_{n=0}^{N-1}[1+\mathrm{i}\delta x e A_\mu(x_n)], \quad (66)$$

where $x_n = n\delta x, \delta x = x/N$. Inserting the complete set,

$$I = \int |\phi,\Lambda\rangle D\mu(\phi)\langle\phi,\Lambda|, \quad (67)$$

we obtain

$$\begin{aligned} W[A] &= \mathrm{tr}\left\{\mathcal{P}\exp\left[\mathrm{i}e\int_0^x A_\mu(x)\mathrm{d}x_\mu\right]\right\} \quad (68)\\ &= \lim_{N\to\infty,\delta x\to 0}\int\cdots\int D\mu(\phi_N)\\ &\times \langle\phi_N,\Lambda|[1+\mathrm{i}\delta x e A_\mu(x_{N-1})]|\phi_{N-1},\Lambda\rangle\\ &\times D\mu(\phi_{N-1})\\ &\times \langle\phi_{N-1},\Lambda|[1+\mathrm{i}\delta x e A_\mu(x_{N-2})]|\phi_{N-2},\Lambda\rangle\\ &\times D\mu(\phi_{N-2})\cdots D\mu(\phi_1)\\ &\times \langle\phi_1,\Lambda|[1+\mathrm{i}\delta x e A_\mu(\phi_0)]|\phi_N,\Lambda\rangle\\ &= \lim_{N\to\infty\delta x\to 0}\prod_{n=1}^{N}\int D\mu(\phi(x_n))\\ &\times\exp\left[\mathrm{i}\delta x\sum_{n=0}^{N-1}e\tilde{A}_\mu(x_n)\right]\\ &\times\prod_{n=0}^{N-1}\langle\phi(x_{n+1}),\Lambda|\phi(x_n),\Lambda\rangle, \end{aligned}$$

[1] For the cases without asymptotic freedom, $b_0^e < 0$, one has $\lambda_0' < 0$ (for example the fermion flavor is larger than 16). The solution of this case, $\lambda = b_0^e e^2/(b_0^\lambda)$ is not consistent and there is no consistent solution at all. As a result the Yang–Mills field is a pure perturbative field

where we have used $\phi_0 = \phi_N$ and we have defined

$$\tilde{A}_\mu(x_n) = \frac{\langle \phi_{n+1}, \Lambda | A_\mu(x_n) | \phi_n, \Lambda\rangle}{\langle \phi_{n+1}, \Lambda | \phi_n, \Lambda\rangle}. \quad (69)$$

Up to $O(\delta x^2)$, we find

$$\begin{aligned}\tilde{A}_\mu(x_n) &= \langle \phi_n, \Lambda | A_\mu(x_n) | \phi_n, \Lambda\rangle + O(\delta x^2) \quad (70)\\ &= \langle \Lambda | U(x_n)^\dagger A(x_n) U(x_n) | \Lambda\rangle + O(\delta x^2),\end{aligned}$$

and

$$\begin{aligned}\langle \phi_{n+1}, \Lambda | \phi_n, \Lambda\rangle &= \langle \phi(x_n), \Lambda | \phi(x_n), \Lambda\rangle \\ &+ \delta x \langle \mathrm{d}U(x_n), \Lambda | \phi(x_n), \Lambda\rangle \\ &+ O(\delta x^2) \quad (71)\\ &= \exp[-\mathrm{i}\delta x \langle \Lambda | \mathrm{i}\mathrm{d}U(x_n)^\dagger U(x_n) | \Lambda\rangle \\ &+ O(\delta x^2)] \\ &= \exp[\mathrm{i}\delta x \langle \Lambda | \mathrm{i}U(x_n)^\dagger \mathrm{d}U(x_n) | \Lambda\rangle \\ &+ O(\delta x^2)].\end{aligned}$$

Thus we obtain the path-integral representation of the gauge field,

$$\begin{aligned}W[A] &= \lim_{N\to\infty, \delta x\to 0} \prod_{n=1}^{N} \int D\mu(\phi(x_n)) \quad (72)\\ &\times \exp\left\{\mathrm{i}e\delta x \sum_{n=0}^{N-1} \langle \Lambda | [U(x_n)^\dagger A(x_n) U(x_n)\right. \\ &+ \mathrm{i}e^{-1} U(x_n)^\dagger \mathrm{d}U(x_n)] | \Lambda\rangle \} \\ &= \int D\mu(\phi) \\ &\times \exp\left(\mathrm{i}e \int_0^x \langle \Lambda | \left[U^\dagger A_\mu U + \frac{\mathrm{i}}{e} U^\dagger \mathrm{d}U\right] | \Lambda\rangle\right) \\ &\times \int D\mu(\phi) \\ &\times \exp\left(\int_0^x \left[\mathrm{i}e\phi^3 \cdot A_\mu^\phi + \mathrm{i}\varpi_\mu\right] \mathrm{d}x_\mu\right),\end{aligned}$$

where

$$\begin{aligned}\varpi_\mu(x) &= \langle \Lambda | \mathrm{i}U^\dagger(x) \partial_\mu U(x) | \Lambda\rangle \quad (73)\\ &= \langle \Lambda | \mathrm{i}U^{-1}(x) \partial_\mu U(x) | \Lambda\rangle\end{aligned}$$

is called the frame connection field. The frame connection field $\varpi_\mu$ is a new U(1) gauge field in two dimensions.

To simplify, we denote the frame connection field by

$$\varpi_\mu(x) = \mathrm{i}U^{-1}(x) \partial_\mu U(x). \quad (74)$$

Accordingly, the gauge field $\tilde{A}_\mu^\phi(x)$ without fixed frame turns into $A_\mu^\phi(x) + (1/e)\varpi_\mu$ where

$$A_\mu^\phi(x) = 2\mathrm{tr}\{A_\mu(x) \boldsymbol{\phi}(x)\} \cdot \boldsymbol{\phi}(x) = \boldsymbol{A}_\mu^\phi \cdot \boldsymbol{\phi}(x). \quad (75)$$

$\boldsymbol{A}_\mu^\phi$ is the image of the gauge field $A_\mu(x)$ projected into the original U(1) gauge manifold without a fixed frame. The gauge transformation of this gauge field is $g(x) = \exp\left[\mathrm{i}\phi^3 \cdot \varphi^3(x)\right]$ for a given frame $\phi^3$. The $\phi^3$-direction covariant derivative operator $\hat{D}_\mu$ of the gauge theory without a fixed frame is

$$\hat{D}_\mu^\phi = \hat{\partial}_\mu + \mathrm{i}eA_\mu^\phi + \mathrm{i}\varpi_\mu. \quad (76)$$

The original U(1) gauge field strength turns into

$$\begin{aligned}\tilde{F}_{\mu\nu}^\phi(x) &\equiv \frac{1}{\mathrm{i}e}\left([\hat{D}_\mu^\phi, \hat{D}_\nu^\phi] - [\hat{\partial}_\mu, \hat{\partial}_\nu]\right) \quad (77)\\ &= \partial_\mu \tilde{A}_\nu^\phi(x) - \partial_\nu \tilde{A}_\mu^\phi(x) + \mathrm{i}e[\tilde{A}_\mu^\phi(x), \tilde{A}_\nu^\phi(x)].\end{aligned}$$

Based on this, the gauge field strength is defined by

$$F_{\mu\nu}(A_\mu^\phi) = \partial_\mu A_\nu^\phi(x) - \partial_\nu A_\mu^\phi(x). \quad (78)$$

The original U(1) gauge field obtains another U(1) local symmetry and turns into U(1) × U(1) gauge theory – one group element is $\exp(\mathrm{i}\delta\varphi^3\phi^3)$ with fixed frame $\phi^3$; the other group gauging the frame $\phi^3$. The operator $L$ transforms $\boldsymbol{\phi}(x)$ to another $\boldsymbol{\phi}'(x)$:

$$L\left[\phi^3(x)\right] = U^{-1}\phi^3(x)\, U = \phi^{3\prime}(x). \quad (79)$$

The gauge field strength is just the curvature $R_{\mu\nu}$,

$$R_{\mu\nu} = \partial_\mu \varpi_\nu - \partial_\nu \varpi_\mu. \quad (80)$$

Because $\phi^3(x) = U^{-1}(x) T^3 U(x)$, the frame connection field $\varpi$ becomes the Maurer–Cartan form as a pure gauge

$$R_{\mu\nu}\left[U^{-1}(x) \partial_\mu U(x)\right] = 0, \quad (81)$$

while for topological non-trivial gauge transformations $U(x)$, one has a non-zero curvature $R_{\mu\nu} \neq 0$.

### 4.2 U(1) gauge field induced by quantum fluctuations of gauge modes

Because the frame may fluctuate, we replace the fixed frame $T^a$ by a frame field $\boldsymbol{\phi}$ and $\exp\left[\mathrm{i}T^1\varphi^1(x) + \mathrm{i}T^2\varphi^2(x)\right]$ by $\exp\left[\mathrm{i}\phi^1\varphi^1(x) + \mathrm{i}\phi^2\varphi^2(x)\right]$. The gauge modes

$$g = g_0 \exp\left[\mathrm{i}\boldsymbol{\phi} \cdot \boldsymbol{\varphi}\right] = g_0 \exp\left[\mathrm{i}\phi^1\varphi^1(x) + \mathrm{i}\phi^2\varphi^2(x)\right] \quad (82)$$

are also divided into two parts: one is the slowly varying vector $g_0$; the other the fast one, $\boldsymbol{\varphi}(x)$. By $g_0$ and $\boldsymbol{\varphi}(x)$, we can rewrite the action

$$\begin{aligned}S_{\mathrm{gf}} = -\frac{1}{2\lambda} \int \mathrm{d}x^2 \mathrm{tr}&\left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^\phi g\right)^2\right.\\ &\left. + \mathrm{i}\alpha\mathcal{C}^\pm(x)\overline{\mathcal{C}}^\pm(x)\right] \quad (83)\end{aligned}$$

as

$$\begin{aligned}S_{\mathrm{gf}} = S\left(g_0, A_\mu^\phi\right) &+ \frac{1}{2\lambda} \int \mathrm{d}x^2 \mathrm{tr}\left\{\left[\partial_\mu\left(\boldsymbol{\phi} \cdot \boldsymbol{\varphi}\right)\right]^2\right. \quad (84)\\ &\left. + \left(g_0^{-1}\partial_\mu g_0 + \mathrm{i}eg_0^{-1}A_\mu^\phi g_0\right)\left[\boldsymbol{\phi} \cdot \boldsymbol{\varphi}, \partial_\mu\left(\boldsymbol{\phi} \cdot \boldsymbol{\varphi}\right)\right]\right\}.\end{aligned}$$

The relevant terms concerning frame fluctuations become

$$\partial_\mu(\boldsymbol{\phi}\cdot\boldsymbol{\varphi}) = \partial_\mu(\phi^a\varphi^a) = \phi^a\partial_\mu\varphi^a + \partial_\mu\phi^a\varphi^a \\ = \phi^a\partial_\mu\varphi^a + \phi^b\omega_\mu^{ab}\varphi^a, \tag{85}$$

where $a = 1, 2$. The definition of the frame connection is used:

$$\partial_\mu\phi^a = \phi^b\omega_\mu^{ab}. \tag{86}$$

The frame $\phi^a$ is really a set of vierbein fields and $\omega^{ab}$ is the Maurer–Cartan one-form.

From

$$\partial_\mu\phi^a = \partial_\mu U^{-1}T^aU + U^{-1}T^a\partial_\mu U \\ = \left[\phi^a, U^{-1}\partial_\mu U\right], \tag{87}$$

a relation between $U^{-1}\partial_\mu U$ and $\omega_\mu^{ab}$ is obtained as follows:

$$\left[\phi^a, U^{-1}\partial_\mu U\right] = \phi^b\omega_\mu^{ab}. \tag{88}$$

The solution is easy to find:

$$U^{-1}\partial_\mu U = -\mathrm{i}\varpi_\mu = \frac{1}{c_v}\epsilon_{ab3}\phi^3\omega_\mu^{ab}, \tag{89}$$

where $c_v = 2$ is the quadratic Casimir operator in the adjoint representation for the SU(2) Lie group. We rewrite the solution as

$$\phi^a\varpi_\mu = \frac{1}{2}\phi^b\omega_\mu^{ab}. \tag{90}$$

The relevant terms for the frame fluctuations become

$$\partial_\mu(\boldsymbol{\phi}\cdot\boldsymbol{\varphi}) = \partial_\mu(\phi^a\varphi^a) = \phi^a\partial_\mu\varphi^a + \partial_\mu\phi^a\varphi^a \\ = \phi^a\partial_\mu\varphi^a + \phi^b\omega_\mu^{ab}\varphi^a \\ = \phi^a\left[(\partial_\mu + 2\varpi_\mu)\varphi^a\right]. \tag{91}$$

In the disorder phase, the Goldstone modes $\boldsymbol{\varphi}(x)$ have a mass gap $m^2$; the relevant terms for the frame fluctuations are

$$\frac{1}{2\lambda}\int \mathrm{d}x^2\left[\mid \boldsymbol{\phi}\cdot(\partial_\mu + 2\varpi_\mu)\boldsymbol{\varphi}\mid^2 + m^2\boldsymbol{\varphi}^2\right]. \tag{92}$$

Hence the Goldstone modes of NLSM are bosons with charge 2 in the presence of a gauge field $\varpi_\mu$.

We obtain the expression of the expansion of the renormalization 2-point function $\varpi_\mu\varpi_\nu$ at one-loop order:

$$\frac{4}{(2\pi)^2}\int \mathrm{d}^2k\left[\frac{(k_\mu+2p_\mu)(k_\nu+2p_\nu)}{(k^2+m^2)((k+p)^2+m^2)} \right. \\ \left. -2\delta_{\mu\nu}\frac{1}{k^2+m^2}\right] \\ = \frac{4}{(2\pi)^2}\int \mathrm{d}^2k\int_0^1 \mathrm{d}x \\ \times\left[\frac{(k_\mu+2p_\mu)(k_\nu+2p_\nu) - 2((k+p)^2+m^2)}{(k^2+2pk+p^2x+m^2)^2}\right] \\ = \frac{4}{(2\pi)^2}\int_0^1 \mathrm{d}x\frac{\pi}{[p^2x(1-x)+m^2]}(1-2x)^2 \\ \times(p_\mu p_\nu - \delta_{\mu\nu}p^2). \tag{93}$$

In the low energy limit $p^2 \to 0$, we simplify the above integral to the following one:

$$\frac{4}{(2\pi)^2}\int_0^1 \mathrm{d}x\frac{\pi}{m^2}(1-2x)^2(p_\mu p_\nu - \delta_{\mu\nu}p^2) \\ = \frac{4}{(2\pi)^2}\frac{\pi}{3m^2}(p_\mu p_\nu - \delta_{\mu\nu}p^2). \tag{94}$$

Because of quantum fluctuations of the massive charged scalar field, a kinetic term of the connection field is induced:

$$\frac{1}{\bar{\mathrm{e}}_\varpi^2}\mathrm{tr}\left[(p_\mu p_\nu - \delta_{\mu\nu}p^2)\varpi_\mu\varpi_\nu\right], \tag{95}$$

where the induced coupling constant is

$$\bar{\mathrm{e}}_\varpi^2 = 3\pi m^2. \tag{96}$$

For reasons of the U(1) local symmetry, the corresponding action of the kinetic term is

$$S_{\mathrm{ind}}(\varpi_\mu) = \int \mathrm{d}^2x\left[\frac{1}{4}(\partial_\mu\varpi_\nu - \partial_\nu\varpi_\mu)^2\right]. \tag{97}$$

Correspondingly in the Yang–Mills vacuum the total gluonic vector potential $A_\mu$ splits into two components,

$$\tilde{A}_\mu = A_\mu^\phi + \frac{1}{e}\varpi_\mu. \tag{98}$$

One is the perturbative part $A_\mu^\phi$; the other the nonperturbative $\varpi_\mu$. In this picture, $\varpi_\mu$ is just a Parisi–Sourlas field in lower dimensions which maintains the stochastic picture of the vacuum. In the propagator of the vector field $\varpi_\mu(x)$, a massless pole appears. Hence the frame connection field $\varpi_\mu(x)$ becomes a dynamical gauge field in two dimensions, giving rise to a confining potential.

### 4.3 Confinement potential from the frame connection field

After obtaining the frame connection field $\omega_\mu$, we absorb the unphysical gauge modes $g$ into the gauge field $A_\mu^\phi(x)$ through a gauge transformation $\Omega^{-1}$:

$$\left(1/(\mathrm{ie})g^{-1}\partial_\mu g + g^{-1}A_\mu^\phi(x)g\right)^{\Omega^{-1}} \\ = \left(\left(A_\mu^\phi\right)^\Omega\right)^{\Omega^{-1}} = A_\mu^\phi(x).$$

Now the action is independent of the gauge modes $g$ and we can isolate the integral $\int D\Omega$ in a normalization factor. Because the frame is not a fixed one in the disorder phase, we replace $\hat{D}_\mu$ by $\hat{D}_\mu^\phi$, $A_\mu^a(x)$ by $\tilde{A}_\mu^\phi(x)$ and $F_{\mu\nu}^a(x)$ by $\tilde{F}_{\mu\nu}^\phi(x)$.

Originating from the term $\tilde{F}_{\mu\nu}^\phi(x)\tilde{F}^{\phi\mu\nu}(x)$, there appears a four-dimensional kinetic term of the frame connection field,

$$S_4(\varpi_\mu) = \int \mathrm{d}x^4\left[-\frac{1}{2}(\partial_\mu\varpi_\mu - \partial_\nu\varpi_\mu)^2\right]. \tag{99}$$

Since $\varpi_\mu$ lies in two dimensions, the above term should be absorbed into $S_{\text{ind}}(\varpi_\mu)$ as follows:

$$\begin{aligned} &S(\varpi_\mu) \\ &= S_{\text{ind}}(\varpi_\mu) + S_4(\varpi_\mu) \\ &= S_{\text{ind}}(\varpi_\mu) + \int dx^4 \left[-\frac{1}{2}(\partial_\mu \varpi_\mu - \partial_\nu \varpi_\mu)^2\right] \\ &= S_{\text{ind}}(\varpi_\mu) + \int dx^2 \left[-\frac{1}{2}\int dz^2 (\partial_\mu \varpi_\mu - \partial_\nu \varpi_\mu)^2\right] \\ &= \int d^2x \left[\frac{1}{2\bar{e}^2_\varpi}(\partial_\mu \varpi_\mu - \partial_\nu \varpi_\mu)^2\right], \end{aligned} \tag{100}$$

with

$$\frac{1}{2\tilde{e}^2_\varpi} = \frac{1}{2\bar{e}^2_\varpi} - \frac{1}{2}\int dz^2 = \frac{1}{2\bar{e}^2_\varpi}. \tag{101}$$

In the above equation, we have used

$$\int dz^2 \equiv 0. \tag{102}$$

Because of the Osp(4|2) supersymmetry, the frame connection field $\varpi_\mu$ becomes a dynamical Parisi–Sourlas Yang–Mills field $\mathcal{W}_l = \mathcal{W}_l(x_\mu, \theta, \bar{\theta})$ after dimensional recovering [15,27]. The external sources of the dynamical Parisi–Sourlas Yang–Mills field $\mathcal{W}_l$ are restricted to a two-dimensional space. The action can be rewritten as an effective model in superspace $\mathcal{M}$,

$$\int d^4x \int \frac{\gamma}{4\pi} d\theta d\bar{\theta} \left[\frac{1}{2\bar{e}^2_\varpi}(\partial_l \mathcal{W}_m - \partial_l \mathcal{W}_m)^2\right], \tag{103}$$

with $l, m = \mu, \theta, \bar{\theta}$.

Finally we arrive at the following dimensional mixing Lagrangian

$$\begin{aligned} \mathcal{L}_f &= \prod_{i=1}^{N_f} \bar{\psi}_i \left[ i\gamma^\mu \left(\hat{\partial}_\mu + ieA^\phi_\mu + i\mathcal{W}_\mu\right) - M_i \right] \psi_i, \quad (104) \\ \mathcal{L}_A &= -\frac{1}{2e^2} \mathrm{tr} F_{\mu\nu}\left(A^\phi_\mu\right) F^{\mu\nu}\left(A^\phi_\mu\right) - \frac{\mathrm{tr}\left(\partial_\mu A^\phi_\mu\right)^2}{2\alpha} \\ &\quad - \mathrm{tr}\overline{\mathcal{C}}^\phi M \mathcal{C}^\phi, \\ \mathcal{L}_{\mathcal{W}} &= \int \frac{\gamma}{4\pi} d\theta d\bar{\theta} \left[\frac{1}{2\bar{e}^2_\varpi}(\partial_l \mathcal{W}_m - \partial_l \mathcal{W}_m)^2\right], \\ \mathcal{L}_{int} &= -\frac{1}{2e^2} \mathrm{tr} \tilde{F}^\phi_{\mu\nu} \tilde{F}^{\mu\nu\phi} + \frac{1}{2e^2} \mathrm{tr} F_{\mu\nu}\left(A^\phi_\mu\right) F^{\mu\nu}\left(A^\phi_\mu\right) \\ &\quad + \frac{1}{2}(\partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu)^2. \end{aligned}$$

Remember the fact that quantum fluctuations of the $\mathcal{W}_\mu$ field lies in two-dimensional superspace while all other fields are in four dimensions. The dimensional constraining condition must be noted that for all Feynman diagrams the internal lines in two-dimensional superspace cannot connect to the four-dimensional internal lines. This condition is very important, for it ensures that the frame connection field $\mathcal{W}_\mu$ does not affect the perturbation behavior of the Yang–Mills theory.

Feynman diagrams for quark–anti-quark scattering by exchanging one gluon and one $\mathcal{W}_\mu$ show the interaction between them (massive or massless). It is not difficult to obtain the potential between quark–anti-quark (massive or massless) for QCD as

$$V(r) = -\frac{1}{4\pi}\frac{e^2}{r} - V_{\text{frame}}(r), \tag{105}$$

where

$$V_{\text{frame}}(r) = -\frac{3\pi\Lambda^2}{2}\exp\left(-\frac{1}{b_0^e e^2}\right) r. \tag{106}$$

This linear potential is equivalent to a string tension because

$$\sigma_0 = \frac{3\pi\Lambda^2}{2}\exp\left(-\frac{1}{b_0^e e^2}\right). \tag{107}$$

This result shows that in the weak coupling region the string tension obeys the scaling law suggested from the result of the renormalization group based on the loop calculations.

Because the boson field $z$ has charge 1 of the frame connection field $-i\varpi_\mu$ (in Appendix A), the linear potential $V_{\text{frame}}(r)$ will be screened by it. Just for this reason we consider $V_{\text{frame}}(r)$ as a short range potential $V_{\text{frame}}(r) = V_{\text{SR}}(r)$. Hence in the limit $r \to \infty$, the confining force exists only for non-integer external charges (in units of $e$). When $q/e$ is an integer, the external charges are screened, $q/e$ not integer, and such screening is incomplete, leaving behind a long range confining potential

$$V_{\text{LR}}(r) = -f\left(\frac{q}{e}\right)\Lambda^2 \exp\left(-\frac{1}{b_0^e e^2}\right) r \tag{108}$$

in the limit $r \to \infty$, where $f(q/e)$ is a function of $q/e$ with a period 1. The periodic behavior of the confining potential with external charges $q$ is found to be

$$V_{\text{LR}}(r, q/e) = V_{\text{LR}}(r, q/e \pm 1). \tag{109}$$

The function $f(q/e)$ is calculated in the next section.

## 5 Source of confinement from magnetic monopoles

### 5.1 Relation between instanton of two-dimensional NLSM and the monopole of four-dimensional gauge theory in abelian gauge

To obtain the O(3) NLSM, we use the adjoint representation,

$$g^\dagger(x) T^3 g(x) = \phi^3 = \boldsymbol{n}(x) = n^a(x) T^a, \tag{110}$$

and

$$n^a(x) = 2\mathrm{tr}(\phi^3 T^a). \tag{111}$$

Let $A_\mu^\phi = 0$, $\mathcal{C}^\pm(x) = \overline{\mathcal{C}}^\pm(x) = 0$; the gauge fixing action turns into the familiar O(3) NLSM

$$\begin{aligned} S_{\text{gf}} &= -\frac{1}{2\lambda}\int \mathrm{d}x^2 \mathrm{tr}\left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu^\phi g\right)^2\right. \\ &\quad \left. +\ \mathrm{i}\alpha\mathcal{C}^\pm(x)\overline{\mathcal{C}}^\pm(x)\right], \qquad (112) \\ &\to -\frac{1}{2\lambda}\int \mathrm{d}x^2 \\ &\times\ \mathrm{tr}\left[g^{-1}(x)\,\partial_\mu g(x)\right]^2 (g \propto \mathrm{SU}(2)/\mathrm{U}(1)) \\ &= \frac{1}{2\lambda}\int \mathrm{d}^2x\,[\partial_\mu \boldsymbol{n}(x)]^2 . \end{aligned}$$

The O(3) NLSM in two dimensions has instanton and anti-instanton solutions, because of the non-trivial homotopy group,

$$\Pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = Z.$$

The instanton is characterized by the Pontryagin index (winding number) defined by

$$Q = \frac{1}{8\pi}\int \mathrm{d}^2x \epsilon_{\mu\nu}\boldsymbol{n}\cdot(\partial_\mu\boldsymbol{n}\times\partial_\nu\boldsymbol{n}). \qquad (113)$$

The action has a lower bound,

$$S_{\text{gf}} = \frac{1}{2\lambda}\int \mathrm{d}^2x\,[\partial_\mu\boldsymbol{n}(x)\cdot\partial_\mu\boldsymbol{n}(x)] \geq \frac{4\pi}{\lambda}|Q|. \qquad (114)$$

The Euclidean action of NLSM is minimized when the above inequality is saturated if

$$\partial_\mu\boldsymbol{n} = \pm\epsilon_{\mu\nu}\boldsymbol{n}\times\partial_\nu\boldsymbol{n}. \qquad (115)$$

Let us introduce a complex field $\boldsymbol{w}$ by a steregraphic projection from the north pole,

$$w_1(x) = \frac{n_1(x)}{1-n_3(x)}, \quad w_2(x) = \frac{n_2(x)}{1-n_3(x)}; \qquad (116)$$

the instanton equation can be rewritten as

$$\partial_1 w = \mp\mathrm{i}\partial_2 w, \quad \boldsymbol{w} = w_1 + \mathrm{i}w_2. \qquad (117)$$

This is equivalent to the Cauchy–Riemann equation,

$$\partial_{\bar z}\boldsymbol{w}(z) = 0, \quad z = x_1 + \mathrm{i}x_2. \qquad (118)$$

This function must not only be analytic, but also meromorphic, since otherwise $\boldsymbol{n}$ would have a branch cut. A typical instanton solution with topological charge $Q$ is given by

$$w(z) = [(z-z_0)/\rho]^Q, \qquad (119)$$

where the constants $\rho$ and $z_0$ are regarded as the size and location of the instanton solution. The one instanton solution implies the solution for the $O(3)$ vector

$$\begin{aligned} n_1 &= \frac{2\rho x_1}{|z-z_0|^2+\rho^2}, \qquad (120) \\ n_2 &= \frac{2\rho x_2}{|z-z_0|^2+\rho^2}, \\ n_3 &= \frac{|z-z_0|^2-\rho^2}{|z-z_0|^2+\rho^2}. \end{aligned}$$

The action of NLSM is written as

$$S_{\text{gf}} = S_c = \frac{4\pi}{\lambda}Q = \frac{1}{b_0^e e^2}Q. \qquad (121)$$

An important fact is that instantons of NLSM can be identified with magnetic monopoles of gauge theory. The magnetic monopole current is

$$K_\mu = -\frac{1}{2e}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial_\nu n^a\partial_\rho n^b\partial_\sigma n^c, \qquad (122)$$

and the magnetic monopole topological charge is defined as

$$\begin{aligned} g_m &= \frac{1}{4\pi}\int \mathrm{d}^3x K_0 \qquad (123) \\ &= -\frac{1}{4\pi}\int \mathrm{d}^2\sigma_i\frac{\mathrm{i}}{2e}\epsilon_{ijk}\boldsymbol{n}\cdot(\partial_j\boldsymbol{n}\times\partial_k\boldsymbol{n}) \\ &= \frac{1}{e}Q. \end{aligned}$$

If $\mu,\nu$ is restricted to two dimensions, $Q$ is the Pontryagin index (winding number) in the NLSM in two-dimensional space $S^2 = \boldsymbol{R}^2\cup\{\infty\}$,

$$Q = \frac{1}{8\pi}\int \mathrm{d}^2x\epsilon_{\mu\nu}\boldsymbol{n}\cdot(\partial_\mu\boldsymbol{n}\times\partial_\nu\boldsymbol{n}). \qquad (124)$$

It is reasonable to assume that the behaviors of the instantons in the two-dimensional NLS model can reflect those of the magnetic monopoles. The contributions from the magnetic monopoles are replaced by those from the instantons in two-dimensional NLSM.

### 5.2 Area law for Wilson loop originating from the magnetic monopole

Knowing the relation of the instanton in two-dimensional NLSM with the magnetic monopole, we can evaluate the Wilson loop expectation value from the contributions from the magnetic monopoles which can be replaced by those from the instantons in two-dimensional NLSM.

The diagonal Wilson loop operator is defined as

$$\begin{aligned} \langle W^{\mathcal{C}}[\mathcal{A}]\rangle &= \frac{1}{\mathcal{N}}\mathrm{tr}\left[\mathcal{P}\exp\left(\mathrm{i}q\oint_{\mathcal{C}}\tilde{A}_\mu^\phi(x)\mathrm{d}x^\mu\right)\right] \qquad (125) \\ &= \frac{1}{\mathcal{N}}\int D\mu(\phi) \\ &\times \exp\left[\int\left(\mathrm{i}q\phi^3\cdot\boldsymbol{A}_\mu^\phi + \mathrm{i}\frac{q}{e}\varpi_\mu\right)\mathrm{d}x_\mu\right], \end{aligned}$$

with the normalization factor $\mathcal{N}^{-1}$. For the planar diagonal Wilson loop $\mathcal{C}$, it can be shown that the vector potential $\tilde{A}_\mu^\phi(x)$ includes the two-dimensional component $\varpi_\mu(x)$.

Considering only the nonperturbative part $\varpi_\mu(x)$, we choose the loop $\mathcal{C}$ contained in the two-dimensional space.

The line integral in the diagonal Wilson loop is rewritten as

$$W^{\mathcal{C}}[\varpi_\mu] = \frac{1}{\mathcal{N}} \int D\mu(\phi) \exp\left[\mathrm{i}\frac{q}{e}\oint_{\mathcal{C}} \varpi_\mu \mathrm{d}x_\mu\right]. \tag{126}$$

According to the Stokes theorem, it is equal to

$$W^{\mathcal{C}}[\varpi_\mu] = \exp\left(\frac{2\pi \mathrm{i}q}{e}\left[\frac{e}{4\pi}\int_S \varepsilon_{\mu\nu} R_{\mu\nu}(x)\mathrm{d}S\right]\right) \tag{127}$$

for any surface $S$ with a boundary $\mathcal{C}$. The diagonal Wilson loop in two-dimensional O(3) NLSM is rewritten as

$$W^{\mathcal{C}}[\varpi_\mu] = \exp\left[\mathrm{i}\frac{2\pi q}{e}\int_S \mathrm{d}^2x \frac{1}{8\pi}\epsilon_{\mu\nu}\boldsymbol{n}\cdot(\partial_\mu \boldsymbol{n}\times\partial_\nu \boldsymbol{n})\right], \tag{128}$$

which is just the instanton number density. This implies that the Wilson loop $W^{\mathcal{C}}[\varpi_\mu]$ counts the number of instanton–anti-instantons existing in the area $S$ bounded by the loop $\mathcal{C}$ in O(3) NLSM.

Remember that the vacuum of NLSM is a kind of $\theta$ vacuum defined as

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}n\theta}|n\rangle. \tag{129}$$

In this paper the theta angle is zero, $\theta = 0$, and the vacuum reads

$$|\theta = 0\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle. \tag{130}$$

The action with a topological angle $\theta$ is written

$$\begin{aligned} S^\theta_{\mathrm{gf}} &= S_{\mathrm{gf}} - \mathrm{i}\theta Q = (n_+ + n_-)S_c \\ &\quad - \mathrm{i}\theta(n_+ - n_-), \\ S_c &= \frac{4\pi}{\lambda} = \frac{1}{b_0^e e^2}. \end{aligned} \tag{131}$$

$n_+$ is the instanton number and $n_-$ the anti-instanton number. In the $\theta$ vacuum the Wilson loop expectation value is

$$\begin{aligned} \langle W^{\mathcal{C}}[\varpi_\mu]\rangle &= \frac{\int \mathrm{d}\mu(\boldsymbol{n})\delta(\boldsymbol{n}\cdot\boldsymbol{n}-1)\mathrm{e}^{-S_{\mathrm{gf}}}W^{\mathcal{C}}[\varpi_\mu]}{\int \mathrm{d}\mu(\boldsymbol{n})\delta(\boldsymbol{n}\cdot\boldsymbol{n}-1)\mathrm{e}^{-S_{\mathrm{gf}}}} \\ &= \frac{Z^{\theta=2\pi q}_{\mathcal{C}}}{Z^{\theta=0}}. \end{aligned} \tag{132}$$

$Z^{\theta=0}$ is the denominator defined by

$$Z^{\theta=0} = \langle\theta=0|\mathrm{e}^{-HT}|\theta=0\rangle. \tag{133}$$

On the other hand, $Z^{\theta=2\pi q}_{\mathcal{C}}$ is the numerator defined by

$$Z^{\theta=2\pi q}_{\mathcal{C}} = \left\langle \begin{matrix}\theta = 2\pi q \text{ inside loop } \mathcal{C} \\ \theta = 0 \text{ outside loop } \mathcal{C}\end{matrix}\right| \mathrm{e}^{-HT} \left| \begin{matrix}\theta = 2\pi q \text{ inside loop } \mathcal{C} \\ \theta = 0 \text{ outside loop } \mathcal{C}\end{matrix}\right\rangle. \tag{134}$$

The calculation of the numerator $Z^{\theta=2\pi q}_{\mathcal{C}}$ reduces to the construction of a system in a $\theta = 0$ vacuum outside the loop and that in a $\theta = 2\pi q$ vacuum inside the loop.

In the dilute-gas approximation, the calculation of the tunneling amplitude is reduced to that of a single instanton contribution $n \to n+1$. The term with $n_+ = 1, n_- = 0$ is given by

$$\begin{aligned} \langle n = \pm 1|\mathrm{e}^{-HT}|0\rangle &= \int \mathrm{d}\mu(\rho)\int \mathrm{d}^2x \exp(-S_c)\exp(\pm\mathrm{i}\theta) \\ &= BL\tau\exp(-S_c), \end{aligned} \tag{135}$$

where $L\tau$ is the (finite but large) volume of two-dimensional space and $B$ is a normalization constant [28–32]. The tunneling amplitude of instantons $\langle n|\mathrm{e}^{-HT}|0\rangle$ is related to the number of well-separated instantons $n_+$ and anti-instantons $n_-$ such that $Q = n = n_+ - n_-$ [20]. All configurations with $n_+$ instantons and $n_-$ anti-instantons must be summed over.

In the dilute-gas approximation, the denominator $Z^{\theta=0}$ is calculated to be

$$\begin{aligned} Z^{\theta=0} &= \langle\theta=0|\mathrm{e}^{-HT}|\theta=0\rangle \\ &= \sum_{n_+,n_-=0}^{\infty} \frac{(BL\tau)^{n_++n_-}}{n_+!n_-!}\exp\left[-(n_++n_-)S_c\right] \\ &= \exp\left[2(BL\tau)\mathrm{e}^{-S_c}\right], \end{aligned} \tag{136}$$

where there is no constraint on the integers $n_+$ or $n_-$, since we are summing over all $Q = n_+ - n_-$.

In the dilute-gas approximation, the numerator is

$$\begin{aligned} Z^{\theta=2\pi q}_{\mathcal{C}} &= \sum_{n_+^{\mathrm{in}},n_-^{\mathrm{in}}=0}^{\infty} \frac{(B|\mathrm{Area}(\mathcal{C})|)^{n_+^{\mathrm{in}}+n_-^{\mathrm{in}}}}{n_+^{\mathrm{in}}!n_-^{\mathrm{in}}!} \\ &\quad\times \mathrm{e}^{-(n_+^{\mathrm{in}}+n_-^{\mathrm{in}})S_c+\mathrm{i}(2\pi qg)(n_+^{\mathrm{in}}-n_-^{\mathrm{in}})} \\ &\quad\times \sum_{n_+^{\mathrm{out}},n_-^{\mathrm{out}}=0}^{\infty} \frac{(B(L\tau-|\mathrm{Area}(\mathcal{C})|))^{n_+^{\mathrm{out}}+n_-^{\mathrm{out}}}}{n_+^{\mathrm{out}}!n_-^{\mathrm{out}}!} \\ &\quad\times \mathrm{e}^{-(n_+^{\mathrm{out}}+n_-^{\mathrm{out}})S_c} \\ &= \exp\Big\{2B \\ &\quad\times\left[|\mathrm{Area}(\mathcal{C})|\cos\frac{2\pi q}{e} + (L\tau - |\mathrm{Area}(\mathcal{C})|)\right] \\ &\quad\times \mathrm{e}^{-S_c}\Big\}. \end{aligned} \tag{137}$$

$|\mathrm{Area}(\mathcal{C})|$ is the area enclosed by the loop $\mathcal{C}$.

In the $\theta$ vacuum of NLSM, the Wilson loop expectation value is

$$\begin{aligned} \langle W^{\mathcal{C}}[\varpi_\mu]\rangle &= \frac{I^{\theta=2\pi q}_{\mathcal{C}}}{I^{\theta=0}} = \exp\Big\{-2B\mathrm{e}^{-S_c} \\ &\quad\times\left(1-\cos\frac{2\pi q}{e}\right)|\mathrm{Area}(\mathcal{C})|\Big\}, \end{aligned} \tag{138}$$

where the action $S_c = 4\pi/\lambda = 1/(b_0^e e^2)$. The Wilson loop integral exhibits the area law. When $q$ is not an integral multiple of the elementary charge $e$, the static quark potential $V_{\text{monop}}(r)$ is given by the linear potential with string tension $\sigma$,

$$V_{\text{monop}}(r) = -\sigma r,$$
$$\sigma = 2B\left(1-\cos\frac{2\pi q}{e}\right)\exp\left(-\frac{1}{b_0^e e^2}\right). \quad (139)$$

It is obvious that $V_{\text{monop}}(r)$ is the long range linear potential for it dominates the screening effect – when $q = Ne$ ($N$ is an integer), the linear potential vanishes. Let $V_{\text{monop}}(r) = V_{\text{LR}}(r)$,

$$V_{\text{LR}}(r) = V_{\text{monop}}(r) = -f\left(\frac{q}{e}\right)\Lambda^2\exp\left(-\frac{1}{b_0^e e^2}\right)r$$
$$= -2B\left(1-\cos\frac{2\pi q}{e}\right)\exp\left(-\frac{1}{b_0^e e^2}\right)r. \quad (140)$$

The periodic function $f(q/e)$ is obtained as

$$f\left(\frac{q}{e}\right) = \frac{2B}{\Lambda^2}\left(1-\cos\frac{2\pi q}{e}\right). \quad (141)$$

## 6 Confinement potential for quark–anti-quark

We have seen in the above sections that the confining potential is a kind of nonperturbative quantum effect, not only from the monopole (or the instanton of NLSM in two dimensions), but also the induced Coulomb potential. But the short range linear potential $V_{\text{SR}}(r)$ (unscreened potential) is related to $V_{\text{frame}}(r)$,

$$V_{\text{SR}}(r) = V_{\text{frame}}(r) = -\frac{q}{e}\frac{3\pi\Lambda^2}{2}\exp\left(-\frac{1}{b_0^e e^2}\right)r, \quad (142)$$

and the long range linear potential $V_{\text{LR}}(r)$ to $V_{\text{monop}}(r)$,

$$V_{\text{LR}}(r) = V_{\text{monop}}(r)$$
$$= -2B\left(1-\cos\frac{2\pi q}{e}\right)\exp\left(-\frac{1}{b_0^e e^2}\right)r. \quad (143)$$

For quark confinement, because the external charges are $\pm e$, an integral multiple of the elementary charge $e$, in the limit $r\to\infty$, the linear potential is totally screened. Particularly the linear potential from the monopoles vanishes:

$$V_{\text{LR}}(r) = V_{\text{monop}}(R) = -2B\exp\left(-\frac{1}{b_0^e e^2}\right)$$
$$\times\left(1-\cos\frac{2\pi q}{e}\right)r = 0, \quad (144)$$

when $q = \pm e$. Hence the confinement potential for quark and anti-quark is a short range linear potential from frame fluctuations,

$$V_{\text{SR}}(r) = V_{\text{frame}}(r), \quad (145)$$

which has nothing to do with magnetic monopoles.

This remarkable result is consistent with that from the confinement mechanism of frame fluctuations in the Lorentz gauge. The gauge fixing term in the Lorentz gauge of the four-dimensional SU(2) Yang–Mills field is reduced to the two-dimensional $\text{SU(2)}_\text{R}\times\text{SU(2)}_\text{L}$ principal chiral model. However, the two-dimensional principal chiral model obtained by dimensional reduction does not have any instanton solution, since

$$\Pi_2\left[\text{SU(2)}\right] = 0.$$

Remember that the two-dimensional O(3) NLSM has instanton solutions because of

$$\Pi_2\left[\text{SU(2)/U(1)}\right] = Z.$$

In Appendix B the short range linear potential between quark and anti-quark in Lorentz gauge is shown to be

$$V(r) = -\frac{\text{c}_2}{2}\bar{\text{e}}_\varpi^2 r, \bar{\text{e}}_\varpi^2 = \frac{3\pi m^2}{\text{c}_2}. \quad (146)$$

Without instanton solutions corresponding to monopoles in Lorentz gauge, it is reasonable to conclude that the confinement has no direct relation with monopoles.

Finally we draw our conclusions. The nonperturbative contributions $\varpi_\mu$ come not only from the magnetic monopole, but also from frame fluctuations, while only the mechanism with frame fluctuations is related to quark confinement (short range linear potential for integer charges).

## 7 Conclusion

Let us give a summary.

The first step in this paper is PS dimensional reduction. By the PS dimensional reduction, the four-dimensional SU(2) Yang–Mills field in abelian gauge is reduced to the two-dimensional O(3) nonlinear $\sigma$ model by the superspace embedding $(4s^2\gamma) = \text{i}\alpha$. The coupling constant $\lambda$ for the O(3) nonlinear $\sigma$ model is obtained from the $\beta(\Lambda)$ function of $e^2$, and $\alpha$ is independent on the gauge parameter $\alpha$.

The aim of this paper is to calculate the confinement potential from two sources: frame fluctuations and monopoles.

When the frame $\phi^3 = U^{-1}(x)T^3U(x)$ is not fixed in the Yang–Mills vacuum, the gauge field $A_\mu(x)$ turns into $A_\mu^\phi(x) + (1/e)\varpi_\mu$. The frame $T^a$ indeed cannot be regarded as a fixed one and the nonperturbative part $\varpi$ appears as a kind of Parisi–Sourlas field. In the propagator of the vector field $\varpi_\mu(x)$, a massless pole appears. Hence the frame connection field $\varpi_\mu(x)$ becomes a dynamical gauge field in two dimensions, giving rise to a short range confining potential.

On the other hand, the contribution to the linear potential from monopoles is also obtained. Because the instanton configuration in two-dimensional O(3) NLSM can be identified with the magnetic monopole configuration in four dimensions, the planar diagonal Wilson loop in four-dimensional SU(2) Yang–Mills theory in MA gauge is calculated in the two-dimensional O(3) NLSM by making use of the PS dimensional reduction. As a result, the magnetic monopole leads to the area law of the Wilson loop for external particles with non-integer charges.

From the fact that the short range confinement potential $V_{\mathrm{LR}}(r)$ for the quark has nothing to do with $V_{\mathrm{monop}}(r)$ (for the simple reason of the integer charge of the quarks), we draw the conclusion that the confinement mechanism is from the frame fluctuations, not the monopole condensation! This remarkable result in MA gauge is consistent with that from the confinement mechanism of frame fluctuations in Lorentz gauge.

## Appendix A: Relation between O(3) NLSM and CP$^1$ model

The O(3) NLSM is locally isomorphic to the CP$^1$ model with the identification

$$n^a(x) := \frac{1}{2} z_i^*(x)(T^a)_{ij} z_j(x) \quad (i,j=1,2), \tag{A1}$$

or

$$n^1 = \mathrm{Re}(z_1^* z_2), \quad n^2 = \mathrm{Im}(z_1^* z_2), \quad n^3 = \frac{1}{2}(|z_1|^2 - |z_2|^2). \tag{A2}$$

Actually, the following constraint is satisfied: $n^A n^A = (|z_1|^2 + |z_2|^2)^2 = 1$. The map from the CP$^1$ model to O(3) NLSM is identified with a Hopf map $H : S^3 \to S^2$ where $S^3$ denotes the unit three-sphere embedded in $R^4$ by $|z_1|^2 + |z_2|^2 = 1$.

For CP$^1$, one can define a vector field

$$V_\mu(x) = \mathrm{i}\boldsymbol{z}^*(x) \cdot \partial_\mu \boldsymbol{z}(x). \tag{A3}$$

The vector field $V_\mu$ is equivalent to the frame connection field $-\mathrm{i}\varpi_\mu$. So the O(3) NLSM is equivalent to the CP$^1$ model,

$$\begin{aligned} S_{\mathrm{gf}} &= \frac{1}{2\lambda}\int \mathrm{d}^2x\,(\partial_\mu \boldsymbol{n})^2 \\ &= \frac{2}{\lambda}\int \mathrm{d}^2x\,[(\partial_\mu - \mathrm{i}V_\mu)\,\boldsymbol{z}]^2 \\ &= \frac{2}{\lambda}\int \mathrm{d}^2x\,[(\partial_\mu + \varpi_\mu)\,\boldsymbol{z}]^2 . \end{aligned} \tag{A4}$$

## Appendix B: Confinement mechanism of frame fluctuations in Lorentz gauge

The Lorentz type gauge fixing term is written

$$\mathcal{L}_{\mathrm{gf}} = -\mathrm{tr}\left(\nu\partial_\mu A_\mu^\Omega + \frac{\alpha}{2}\nu^2 + \overline{\mathcal{C}}\partial_\mu D_\mu\left[(A_\mu^\Omega)\right]\mathcal{C}\right), \tag{B1}$$

where the gauge field $A_\mu^\Omega$ is gauge dependent on

$$A_\mu^{\Omega=g} = g^{-1}A_\mu g + \frac{1}{\mathrm{i}e}g^{-1}\partial_\mu g.$$

By the BRST $\delta_B$ and anti-BRST $\bar{\delta}_B$ [14–19,13], $\mathcal{L}_{\mathrm{gf}}$ turns into the form

$$\mathcal{L}_{\mathrm{gf}} = \mathrm{i}\delta_B\bar{\delta}_B \mathrm{tr}\left[\frac{1}{2}(A_\mu^\Omega)^2 + \frac{\mathrm{i}\alpha}{2}\mathcal{C}\overline{\mathcal{C}}\right], \tag{B2}$$

which is a contravariant supervector which transforms like the supercoordinate under Osp(4 | 2) [14–19,13].

The derivatives in the direction of $\theta, \bar{\theta}$ proportional to the BRST and the anti-BRST transformations are

$$\frac{\partial}{\partial\theta} = s\delta_B, \quad \frac{\partial}{\partial\bar{\theta}} = s\bar{\delta}_B, \tag{B3}$$

where $s$ is the superspace embedding factor to be determined. One requires $\mathcal{A}_\mu(X)\ |_{\theta=0,\bar{\theta}=0} = \mathcal{A}_\mu(x,0,0) = A_\mu(x)$ and “the superspace embedding relation” is found to be

$$\mathcal{A}_\theta(x) = s\mathcal{C}(x),\ \mathcal{A}_{\bar{\theta}}(x) = s\overline{\mathcal{C}}(x), \tag{B4}$$

$$\begin{aligned} \eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X) = {} & [\mathcal{A}^\mu(X)]^\Omega\,[\mathcal{A}_\mu(X)]^\Omega \\ & + \mathrm{i}\alpha\mathcal{C}(X)\overline{\mathcal{C}}(X). \end{aligned} \tag{B5}$$

From this, the relation between $s$ and $\gamma$ in superspace is obtained as $\left(4s^2\gamma\right) = \mathrm{i}a$.

Then the action is reduced to a two-dimensional sigma model:

$$\begin{aligned} S_{\mathrm{gf}} = {} & -\frac{1}{2\lambda}\int \mathrm{d}x^2 \mathrm{tr}\left[\left(g^{-1}\partial_\mu g + \mathrm{i}eg^{-1}A_\mu g\right)^2\right. \\ & \left. + \mathrm{i}\alpha\mathcal{C}(x)\bar{\mathcal{C}}(x)\right], \end{aligned} \tag{B6}$$

where $\lambda = -b_0^e e^2/b_0^\lambda$.

Two-dimensional $\mathrm{SU(2)_R} \times \mathrm{SU(2)_L}$ principal chiral model has only one phase: a disorder phase without long range order,

$$\langle g\rangle\ |_{2D} = 0. \tag{B7}$$

Goldstone excitons are massive. Up to one loop, the mass gap of the principal chiral model in two dimensions has been obtained in a large $N$ approximation;

$$m^2 = \Lambda^2 \exp\left(-\frac{8\pi}{3\lambda}\right) = \Lambda^2 \exp\left(-\frac{1}{b_0^e e^2}\right), \tag{B8}$$

where $\Lambda$ is a cut-off.

In the disorder phase frame fluctuations are induced by quantum fluctuations of the gauge modes and the frame $T^a$ cannot be regarded as a fixed one. We introduce the frame field $\boldsymbol{\phi}(x) = U^{-1}(x)T^aU(x)$ ($\mathrm{U}(x) \in \mathrm{SU(2)_R} \times \mathrm{SU(2)_L/SU(2)}$) to indicate the color-direction variable. $\boldsymbol{\phi}(x)$ is a composite field of the gauge modes $g(x)$.

Without a fixed frame, the gauge field $A_\mu(x) = A_\mu^a(x)T^a$ changes its path-integral form. Accordingly, the gauge field $\tilde{A}_\mu^\phi(x)$ without fixed frame turns into $A_\mu^\phi(x) + (1/e)\varpi_\mu$ where

$$A_\mu^\phi(x) = 2\mathrm{tr}\{A_\mu(x)\boldsymbol{\phi}(x)\} \cdot \boldsymbol{\phi}(x) = \boldsymbol{A}_\mu^\phi \cdot \boldsymbol{\phi}(x). \tag{B9}$$

$A_\mu^\phi$ is the image of the gauge field $A_\mu(x)$ projected into the original SU(2) gauge manifold without a fixed frame. $\varpi_\mu$ is a new SU(2) gauge field in two dimensions denoted

$$\varpi_\mu(x) = \mathrm{i}U^{-1}(x)\partial_\mu U(x). \tag{B10}$$

The gauge transformation is $g(x) = \exp[\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\varphi}(x)]$ for a given frame $\boldsymbol{\phi}$. The $\boldsymbol{\phi}$-direction covariant derivative operator $\hat{D}_\mu$ of the gauge theory without a fixed frame is

$$\hat{D}_\mu^\phi = \hat{\partial}_\mu + \mathrm{i}eA_\mu^\phi + \mathrm{i}\varpi_\mu. \tag{B11}$$

Hence the original SU(2) Yang–Mills field obtains another SU(2) local symmetry and turns into SU(2) × SU(2) gauge theory – one group element is $\exp(\mathrm{i}\delta\varphi^a\phi^a)$ with fixed frame $\phi^a$; the other group gauging the frame $\phi^a$.

Because the frame may fluctuate, we replace the fixed frame $T^a$ by a frame field $\boldsymbol{\phi}$ and $\exp[\mathrm{i}T^a\varphi^a(x)]$ by $\exp[\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\varphi}(x)]$. In the disorder phase, Goldstone modes $\boldsymbol{\varphi}(x)$ have a mass gap $m^2$, and the relevant terms of Goldstone modes are

$$\frac{1}{2\lambda}\int \mathrm{d}x^2\left[\mid \boldsymbol{\phi}\cdot(\partial_\mu + 2\varpi_\mu)\boldsymbol{\varphi}\mid^2 + m^2\boldsymbol{\varphi}^2\right]. \tag{B12}$$

Hence the Goldstone modes of the principal chiral model are bosons with charge 2 in the presence of a gauge field $\varpi_\mu$. Because of the SU(2) local symmetry, the corresponding action of the kinetic term that is induced is

$$S_{\mathrm{ind}}(\varpi_\mu) = \int \mathrm{d}^2x\left[\frac{1}{2\bar{\mathrm{e}}_\varpi^2}\mathrm{tr}\left(\partial_\mu\varpi_\nu - \partial_\nu\varpi_\mu + [\varpi_\mu, \varpi_\upsilon]\right)^2\right]. \tag{B13}$$

Correspondingly in the QCD vacuum the total gluonic vector potential $A_\mu$ splits into two components:

$$\tilde{A}_\mu = A_\mu^\phi + \frac{1}{e}\varpi_\mu. \tag{B14}$$

One is the perturbative part $A_\mu^\phi$; the other the nonperturbative $\varpi_\mu$. $\varpi_\mu$ is not a superposition of classical solutions like instantons, monopoles etc., but from purely quantum fluctuations. In this picture, $\varpi_\mu$ is just a Parisi–Sourlas field in two dimensions which maintains the stochastic picture of the vacuum. In the propagator of the vector field $\varpi_\mu(x)$, a massless pole appears. Hence the frame connection field $\varpi_\mu(x)$ becomes a dynamical gauge field in two dimensions, giving rise to a confining potential.

Feynman diagrams for quark–anti-quark scattering by exchanging one gluon and one $\varpi_\mu$ show the interaction between them (massive or massless). It is not difficult to obtain the potential between quark and anti-quark (massive or massless) in four dimensions:

$$V(r) = -\frac{c_2}{4\pi}\frac{e^2}{r} - \frac{c_2}{2}e_\omega^2 r, \tag{B15}$$

where

$$\bar{e}_\varpi^2 = \frac{3\pi m^2}{c_2} = \frac{9\pi\Lambda^2}{4}\exp\left[-\frac{1}{b_0^e e^2}\right]. \tag{B16}$$

This linear potential is equivalent to a string tension:

$$\sigma_{\mathrm{frame}} = \frac{3\pi\Lambda^2}{2}\exp\left[-\frac{1}{b_0^e e^2}\right], \tag{B17}$$

which shows the result of the renormalization group based on the loop calculations in the weak coupling region [3].

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